



TITLE:

FURUTA INEQUALITY AND \mathbb{P} - Ω $A(s,t)$ OPERATORS (The research of geometric structures in quantum information based on Operator Theory and related topics)

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FURUTA INEQUALITY AND p - $wA(s, t)$ OPERATORS

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Dedicated to the memory of Professor Takayuki Furuta with deep gratitude

ABSTRACT. The aim of this paper is to introduce small history with Furuta's inequality and relating class of p - $wA(s, t)$ operators.

1. INTRODUCTION

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space \mathcal{H} . In 1987, Furuta [5] proved the following inequality.

Theorem 1. [Furuta inequality]

Let $0 < p, q, r \in \mathbb{R}$ and $A, B \in B(\mathcal{H})$ satisfy $0 \leq B \leq A$. If $p + 2r \leq (1 + 2r)q$ and $1 \leq q$, then $B^{\frac{p+2r}{q}} \leq (B^r A^p B^r)^{\frac{1}{q}}$ and $(A^r B^p A^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}}$.

This is a good extension of Löwner-Heinz's inequality ([7] and [12]).

Theorem 2. [Löwner-Heinz's inequality]

Let $A, B \in B(\mathcal{H})$ satisfy $0 \leq B \leq A$ and $0 < p \leq 1$. Then $B^p \leq A^p$.

Recall that an operator T is said to be hyponormal if $T^*T \geq TT^*$. For $T \in B(\mathcal{H})$, set $|T| = (T^*T)^{\frac{1}{2}}$ as usual. By taking $U|T|x = Tx$ for $x \in \mathcal{H}$ and $Ux = 0$ for $x \in \ker |T|$, T has a unique polar decomposition $T = U|T|$ with $\ker U = \ker |T|$. Aluthge [1] defined Aluthge transformation $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, and studied interesting properties of p -hyponormal operators for $0 < p \leq 1$.

Definition 3. $T \in B(\mathcal{H})$ is said to be p -hyponormal if $(T^*T)^p \geq (TT^*)^p$ where $p \in (0, 1]$.

The class of p -hyponormal operators is a generalization of the class of hyponormal operators by Löwner-Heinz's inequality. Aluthge [1] proved following result.

Theorem 4. Let T be p -hyponormal. If $0 < p \leq 1/2$, then \tilde{T} is $(p + 1/2)$ -hyponormal. If $1/2 \leq p \leq 1$, then \tilde{T} is hyponormal.

This is a epoch making result. Aluthge transformation is a strong tool of operator theory and many applications have been studied, for example, Putnam inequality, Fuglede Putnam type theorem, Wyle type theorem. I think that generalization of class of operators may be a good way to investigate non-normal operators. Furuta [6] and Yoshino [17] defined generalized transformation $T(s, t) = |T|^s U |T|^t$ with $0 < s, t$ and Yanagida [15], Ito [8], Yamazaki [9], Fujii, Jung, Lee, Nakamoto [4] studied $wA(s, t)$ operators.

Definition 5. T is said to be $wA(s, t)$ if

$$(1.1) \quad (|T^*|^t |T|^{2s} |T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$$

and

$$(1.2) \quad |T|^{2s} \geq (|T|^s |T^*|^{2t} |T|^s)^{\frac{s}{s+t}}.$$

Hence generalized Aluthge transformation $T(s, t)$ of $wA(s, t)$ operator T enjoys the following property.

Proposition 6. *Let T be $wA(s, t)$. Then*

$$|T(s, t)|^{\frac{2t}{s+t}} \geq |T|^{2t}$$

and

$$|T|^{2s} \geq |T(s, t)^*|^{\frac{2s}{s+t}}.$$

Hence

$$|T(s, t)|^{\frac{2r}{s+t}} \geq |T|^{2r} \geq |T(s, t)^*|^{\frac{2r}{s+t}}$$

for all $r \in (0, \min\{s, t\}]$.

Ito and Yamazaki [9] proved that (1.1) implies (1.2). This is a good result. This means that class of $wA(s, t)$ operators coincides with $A(s, t)$ operators.

Definition 7. T is said to be $A(s, t)$ if

$$(1.3) \quad (|T^*|^t |T|^{2s} |T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}.$$

Class $A(1, 1)$ is said to be class A and class $A(1/2, 1/2)$ is said to be w -hyponormal [4, 9, 15]. Prasad and Tanahshi [16] defined p - $wA(s, t)$ operator for $0 < p \leq 1$ and $0 < s, t, s + t \leq 1$ as follows.

Definition 8. T is said to be p - $wA(s, t)$ if

$$(1.4) \quad (|T^*|^t |T|^{2s} |T^*|^t)^{\frac{pt}{s+t}} \geq |T^*|^{2pt}$$

and

$$(1.5) \quad |T|^{2ps} \geq (|T|^s |T^*|^{2t} |T|^s)^{\frac{ps}{s+t}}.$$

Hence p - $wA(s, t)$ operator is a generalization of $wA(s, t)$ operator by Löwner-Heinz's inequality. The aim of this paper is to prove several properties of p - $wA(s, t)$ operator and show some open problems of p - $wA(s, t)$ operator. Main results are proved in [16] and [2].

2. RESULTS

At first, we show generalized Aluthge transformation $T(s, t)$ of p - $wA(s, t)$ operator T enjoys the following property [16].

Theorem 9. *Let T be p - $wA(s, t)$. Then*

$$|T(s, t)|^{\frac{2pt}{s+t}} \geq |T|^{2pt}$$

and

$$|T|^{2ps} \geq |T(s, t)^*|^{\frac{2ps}{s+t}}.$$

Hence

$$|T(s, t)|^{\frac{2pr}{s+t}} \geq |T|^{2pr} \geq |T(s, t)^*|^{\frac{2pr}{s+t}}$$

for all $r \in (0, \min\{s, t\}]$.

Proof.

$$\begin{aligned}
& (|T^*|^t |T|^{2s} |T^*|^t)^{\frac{pt}{s+t}} \geq |T^*|^{2pt} \\
& \iff (U|T|^t U^* |T|^{2s} U|T|^t U^*)^{\frac{pt}{s+t}} \geq U|T|^{2pt} U^* \\
& \iff U(|T|^t U^* |T|^{2s} U|T|^t)^{\frac{pt}{s+t}} U^* \geq U|T|^{2pt} U^* \quad ([8, \text{Lemma 2.1}]) \\
& \iff (|T|^t U^* |T|^{2s} U|T|^t)^{\frac{pt}{s+t}} \geq |T|^{2pt} \quad ([8, \text{lemma 2.1}]) \\
& \iff |T(s, t)|^{\frac{2pt}{s+t}} \geq |T|^{2pt}.
\end{aligned}$$

Also,

$$\begin{aligned}
& (|T|^s |T^*|^{2t} |T|^s)^{\frac{ps}{s+t}} \leq |T|^{2ps} \\
& \iff (|T|^s U|T|^{2t} U^* |T|^s)^{\frac{ps}{s+t}} \leq |T|^{2ps} \\
& \iff |\{T(s, t)\}^*|^{\frac{2ps}{s+t}} \leq |T|^{2ps}.
\end{aligned}$$

□

Next we show class of p - $wA(s, t)$ operators are decreasing class of operators with $0 < p \leq 1$ and increasing with $0 < s, t \leq 1$. The proof is essentially due to C. Yang and J. Yuan ([19] Proposition 3.4).

Lemma 10. *If T is p - $wA(s, t)$ and $0 < s \leq s_1, 0 < t \leq t_1, 0 < p_1 \leq p \leq 1$, then T is p_1 - $wA(s_1, t_1)$.*

Proof. Let T be p - $wA(s, t)$. Then

$$(2.1) \quad (|T^*|^t |T|^{2s} |T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}$$

and

$$(2.2) \quad |T|^{2sp} \geq (|T|^s |T^*|^{2t} |T|^s)^{\frac{sp}{s+t}}.$$

We prove that T is p - $wA(s_1, t_1)$. Then T is p_1 - $wA(s_1, t_1)$ by Lowner-Heinz's inequality.

Let $A_1 = (|T^*|^t |T|^{2s} |T^*|^t)^{\frac{tp}{s+t}}$ and $B_1 = |T^*|^{2tp}$. Since (1) implies $A_1 \geq B_1$, we have

$$\left(B_1^{\frac{r_2}{2}} A_1^{p_2} B_1^{\frac{r_2}{2}} \right)^{\frac{1+r_2}{p_2+r_2}} \geq B_1^{1+r_2}$$

for any $r_2 > 0$ and $p_2 \geq 1$ by Furuta's inequality [5]. Let

$$\beta \geq t, p_2 = \frac{s+t}{tp} \geq 1, r_2 = \frac{\beta-t}{tp} \geq 0.$$

Then

$$\left(|T^*|^\beta |T|^{2s} |T^*|^\beta \right)^{\frac{tp+\beta-t}{s+\beta}} \geq |T^*|^{2tp+2\beta-2t}.$$

Hence we have

$$\left(|T^*|^\beta |T|^{2s} |T^*|^\beta \right)^{\frac{w}{s+\beta}} \geq |T^*|^{2w}$$

for any $0 < w \leq tp + \beta - t$.

Let

$$f_s(\beta) = \left(|T|^s |T^*|^{2\beta} |T|^s \right)^{\frac{s}{s+\beta}}$$

for $\beta \geq t$. Then

$$\begin{aligned} f_s(\beta) &= \left\{ \left(|T|^s |T^*|^{2\beta} |T|^s \right)^{\frac{s+\beta+w}{s+\beta}} \right\}^{\frac{s}{s+\beta+w}} \\ &= \left\{ |T|^s |T^*|^\beta \left(|T^*|^\beta |T|^{2s} |T^*|^\beta \right)^{\frac{w}{s+\beta}} |T^*|^\beta |T|^s \right\}^{\frac{s}{s+\beta+w}} \\ &\geq \left\{ |T|^s |T^*|^\beta |T^*|^{2w} |T^*|^\beta |T|^s \right\}^{\frac{s}{s+\beta+w}} \\ &= \left\{ |T|^s |T^*|^{2(\beta+w)} |T|^s \right\}^{\frac{s}{s+\beta+w}} \\ &= f_s(\beta + w). \end{aligned}$$

Hence $f_s(\beta)$ is decreasing for $\beta \geq t$.

Then, by (2.2),

$$\begin{aligned} |T|^{2sp} &\geq \left(|T|^s |T^*|^{2t} |T|^s \right)^{\frac{sp}{s+t}} \\ &= \{f_s(t)\}^p \\ &\geq \{f_s(t_1)\}^p = \left(|T|^s |T^*|^{2t_1} |T|^s \right)^{\frac{sp}{s+t_1}}. \end{aligned}$$

Let $A_2 = |T|^{2sp}$ and $B_2 = \left(|T|^s |T^*|^{2t_1} |T|^s \right)^{\frac{sp}{s+t_1}}$. Then

$$A_2^{1+r_3} \geq \left(A_2^{\frac{r_3}{2}} B_2^{p_3} A_2^{\frac{r_3}{2}} \right)^{\frac{1+r_3}{p_3+r_3}}$$

for any $r_3 \geq 0$ and $p_3 \geq 1$ by Furuta's inequality [5]. Let

$$p_3 = \frac{s+t_1}{sp} \geq 1, r_3 = \frac{s_1-s}{sp} \geq 0.$$

Then

$$|T|^{2sp+2s_1-2s} \geq \left(|T|^{s_1} |T^*|^{2t_1} |T|^{s_1} \right)^{\frac{sp+s_1-s}{s_1+t_1}}.$$

Since

$$sp + s_1 - s - s_1p = (s_1 - s)(1 - p) \geq 0,$$

we have

$$|T|^{2s_1p} \geq \left(|T|^{s_1} |T^*|^{2t_1} |T|^{s_1} \right)^{\frac{s_1p}{s_1+t_1}}.$$

Similarly, we have

$$\left(|T^*|^{t_1} |T|^{2s_1} |T^*|^{t_1} \right)^{\frac{t_1p}{s_1+t_1}} \geq |T^*|^{2t_1p}.$$

Hence T is p - $wA(s_1, t_1)$.

□

The following result seems new, even for class $A(s, t)$ operators.

Theorem 11. *If T is p - $wA(s, t)$ and T is invertible, then T^{-1} is p - $wA(t, s)$.*

Proof. Let $T = U|T|$ the polar decomposition of T . Then

$$|T^{-1}|^2 = (T^{-1})^* T^{-1} = (T^*)^{-1} T^{-1} = (TT^*)^{-1} = |T^*|^{-2}.$$

Hence

$$|T^{-1}| = |T^*|.$$

Also,

$$|(T^{-1})^*|^2 = (T^{-1})(T^{-1})^* = T^{-1}(T^*)^{-1} = (T^*T)^{-1} = |T|^{-2}.$$

Hence

$$|(T^{-1})^*| = |T|^{-1}.$$

Then

$$\begin{aligned} & \{ |(T^{-1})^*|^s |T^{-1}|^{2t} |(T^{-1})^*|^s \}_{\frac{sp}{s+t}} \\ &= (|T|^{-s} |T^*|^{-2t} |T|^{-s})_{\frac{sp}{s+t}} \\ &= (|T|^s |T^*|^{2t} |T|^s)_{\frac{-ps}{s+t}} \\ &\geq |T|^{-2sp} = |(T^{-1})^*|^{2sp} \end{aligned}$$

and

$$\begin{aligned} & \{ |T^{-1}|^t |(T^{-1})^*|^{2s} |T^{-1}|^t \}_{\frac{tp}{s+t}} \\ &= \{ |T^*|^{-t} |T|^{-2s} |T^*|^{-t} \}_{\frac{tp}{s+t}} \\ &= (|T^*|^t |T|^{2s} |T^*|^t)_{\frac{-tp}{s+t}} \\ &\leq |T^*|^{-2tp} = |T^{-1}|^{2tp}. \end{aligned}$$

□

Corollary 12. *If T is $A(s, t)$ and T is invertible, then T^{-1} is $A(t, s)$.*

Let $0 < p \leq 1$ and $S, T \in B(\mathcal{H})$ be non zero operators. In [3], Duggal proved that tensor product $T \otimes S$ is p -hyponormal if and only if T and S are p -hyponormal. The passage of class A operators is studied by Jeon and Duggal [10]. Tanahashi and Chō [13] proved that the tensor product $T \otimes S$ is of class $A(s, t)$ if and only if T and S are class $A(s, t)$. Now we will prove similar result for p -class $wA(s, t)$ operators by adopting the ideas in [13], [10].

Lemma 13. [11] *Let $T_1, T_2, S_1, S_2 \in B(\mathcal{H})$ be non negative operators. If $T_1 \neq 0$ and $S_1 \neq 0$, then the following conditions are equivalent.*

- (1) $T_1 \otimes S_1 \leq T_2 \otimes S_2$.
- (2) *There exists $c > 0$ such that $T_1 \leq cT_2$ and $S_1 \leq c^{-1}S_2$.*

Lemma 14. [13] *Let $T = U|T|$ and $S = V|S|$ be the polar decompositions of $T, S \in B(\mathcal{H})$. Then the following assertions hold.*

- (1) $|T \otimes S| = |T| \otimes |S|$.
- (2) $T \otimes S = (U \otimes V)(|T| \otimes |S|)$ is the polar decomposition of $T \otimes S$.
- (3) $(T \otimes S)(s, t) = T(s, t) \otimes S(s, t)$ for $s, t > 0$.

Theorem 15. *Let $S, T \in B(\mathcal{H})$ be non zero operators. Then $T \otimes S$ is p - $wA(s, t)$ if and only if S, T are p - $wA(s, t)$.*

Proof. Let $S, T \in B(\mathcal{H})$ be non zero p -class $wA(s, t)$ operators and $S = V|S|, T = U|T|$ be polar decompositions of S, T . Then $|T(s, t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$ and $|S(s, t)^*|^{\frac{2tp}{s+t}} \geq |S|^{2tp}$ by Theorem 9. By applying Lemma 14, we obtain

$$\begin{aligned} |(T \otimes S)(s, t)|^{\frac{2tp}{s+t}} &= |T(s, t) \otimes S(s, t)|^{\frac{2tp}{s+t}} \\ &= |T(s, t)|^{\frac{2tp}{s+t}} \otimes |S(s, t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp} \otimes |S|^{2tp} = |T \otimes S|^{2tp}. \end{aligned}$$

Similarly, we have

$$|T \otimes S|^{2sp} \geq |\{(T \otimes S)(s, t)\}^*|^{\frac{2sp}{s+t}}.$$

Hence $T \otimes S$ is p - $wA(s, t)$.

Conversely, suppose that $T \otimes S$ is p - $wA(s, t)$. Then

$$\begin{aligned} |(T \otimes S)(s, t)|^{\frac{2tp}{s+t}} &= |T(s, t)|^{\frac{2tp}{s+t}} \otimes |S(s, t)|^{\frac{2tp}{s+t}} \\ &\geq |T|^{2tp} \otimes |S|^{2tp} = |T \otimes S|^{2tp} \end{aligned}$$

and

$$|T \otimes S|^{2sp} \geq |\{(T \otimes S)(s, t)\}^*|^{\frac{2sp}{s+t}}.$$

Hence there exists $c > 0$ such that

$$c|T(s, t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$$

and

$$c^{-1}|S(s, t)|^{\frac{2tp}{s+t}} \geq |S|^{2tp}$$

by Lemma 13. Let x be a unit vector. Then

$$\begin{aligned} \| |T|^{tp} x \|^2 &= \langle |T|^{2tp} x, x \rangle \leq \langle c|T(s, t)|^{\frac{2tp}{s+t}} x, x \rangle \\ &\leq c \| |T(s, t)|^{\frac{tp}{s+t}} \|^2 = c \| |T(s, t)|^{\frac{tp}{s+t}} \|^2 \\ &= c \| |T|^s U |T|^t \|^{\frac{2tp}{s+t}} \\ &\leq c (\| |T|^s \| \cdot 1 \cdot \| |T|^t \|)^{\frac{2tp}{s+t}} = c \| |T|^{tp} \|^2. \end{aligned}$$

Hence $1 \leq c$. Similarly,

$$\begin{aligned} \| |S|^{tp} x \|^2 &= \langle |S|^{2tp} x, x \rangle \leq \langle c^{-1}|S(s, t)|^{\frac{2tp}{s+t}} x, x \rangle \\ &\leq c^{-1} \| |S(s, t)|^{\frac{tp}{s+t}} \|^2 = c^{-1} \| |S(s, t)|^{\frac{tp}{s+t}} \|^2 \\ &= c^{-1} \| |S|^s V |S|^t \|^{\frac{2tp}{s+t}} \\ &\leq c^{-1} (\| |S|^s \| \cdot 1 \cdot \| |S|^t \|)^{\frac{2tp}{s+t}} = c^{-1} \| |S|^{tp} \|^2. \end{aligned}$$

Hence $1 \leq c^{-1}$. Hence $c = 1$. This implies

$$|T(s, t)|^{\frac{2tp}{s+t}} \geq |T|^{2tp}$$

and

$$|S(s, t)|^{\frac{2tp}{s+t}} \geq |S|^{2tp}.$$

Similarly we have

$$|T|^{2sp} \geq |\{T(s, t)\}^*|^{\frac{2sp}{s+t}}$$

and

$$|S|^{2sp} \geq |\{S(s, t)\}^*|^{\frac{2sp}{s+t}}.$$

Thus T and S are p - $wA(s, t)$. \square

Corollary 16. *Let $S, T \in B(\mathcal{H})$ be non zero operators. Then $T \otimes S$ is p - $A(s, t)$ if and only if S, T are p - $A(s, t)$.*

Theorem 17. *Let $T \in B(\mathcal{H})$ be p - $wA(s, t)$ with $0 < s, t, s + t = 1$ and $0 < p \leq 1$. Let $\rho e^{i\theta} \in \mathbb{C}$ be an isolated point of $\sigma(T)$ and $0 < \rho$. Then the Riesz idempotent E for T with respect to $\rho e^{i\theta}$ is self-adjoint with*

$$\text{ran } E = \ker(T - \rho e^{i\theta}) = \ker\left((T - \rho e^{i\theta})^*\right).$$

and coincides with the Riesz idempotent $E(s, t)$ for $T(s, t)$ with respect to $\rho e^{i\theta}$.

Proof. Since $\sigma(T) = \sigma(T(s, t))$ by Lemma 6 of [14], $\rho e^{i\theta}$ is an isolated point of $\sigma(T(s, t))$. Since $T(s, t)$ is rp -hyponormal for all $r \in (0, \min\{s, t\}]$, $E(s, t)$ is self-adjoint and satisfies

$$\text{ran } E(s, t) = \ker\left(T(s, t) - \rho e^{i\theta}\right) = \ker(T - \rho e^{i\theta})$$

and

$$\rho e^{i\theta} \notin \sigma\left(T(s, t)|_{\text{ran } E(s, t)}\right).$$

Since $\ker(T - \rho e^{i\theta}) = \text{ran } E(s, t)$ reduces T , we have

$$T = \rho e^{i\theta} \oplus T' \text{ on } \mathcal{H} = \text{ran } E(s, t) \oplus \text{ran } (1 - E(s, t)).$$

Then T' is also class p - $wA(s, t)$ and $T'(s, t) = T(s, t)|_{\text{ran } (1 - E(s, t))}$. Hence $\rho e^{i\theta} \notin \sigma(T'(s, t)) = \sigma(T')$ by Lemma 6 of [14]. Hence $T' - \rho e^{i\theta}$ is invertible and $T - \rho e^{i\theta} = 0 \oplus (T' - \rho e^{i\theta})$. This implies $\ker(T - \rho e^{i\theta}) = \ker((T - \rho e^{i\theta})^*)$ and

$$E = \frac{1}{2\pi i} \int_{\gamma} (z - \rho e^{i\theta})^{-1} \oplus (z - T')^{-1} dz = 1 \oplus 0 = E(s, t)$$

where γ is a small circle containing $\rho e^{i\theta}$. \square

Theorem 18. *Let $T \in B(\mathcal{H})$ be p - $wA(s, t)$ with $0 < s, t, s + t \leq 1$ and $0 < p \leq 1$. Let $(T - \rho e^{i\theta})x_n \rightarrow 0$ for $x_n \in \mathcal{H}$ with $\|x_n\| = 1$ and $\rho e^{i\theta} \in \mathbb{C}, 0 < \rho$. Then $(|T| - \rho)x_n, (U - e^{i\theta})x_n, (U - e^{i\theta})^*x_n, (T - \rho e^{i\theta})^*x_n \rightarrow 0$.*

Proof. We may assume $s + t = 1$ by Lemma 10. Since

$$\left(T(s, t) - \rho e^{i\theta}\right) |T|^s x_n = |T|^s \left(T - \rho e^{i\theta}\right) x_n \rightarrow 0,$$

we have

$$\left(T(s, t) - \rho e^{i\theta}\right)^* |T|^s x_n \rightarrow 0,$$

because $T(s, t)$ is rp -hyponormal for all $r \in (0, \min\{s, t\}]$. Hence

$$\begin{aligned} 0 &\leftarrow \left(T(s, t) - \rho e^{i\theta}\right)^* \left(T(s, t) - \rho e^{i\theta}\right) |T|^s x_n \\ &= (|T(s, t)|^2 - \rho^2) |T|^s x_n \\ &\quad - \rho e^{-i\theta} \left(T(s, t) - \rho e^{i\theta}\right) |T|^s x_n - \rho \left(T(s, t) - \rho e^{i\theta}\right)^* |T|^s x_n. \end{aligned}$$

This implies

$$(|T(s, t)|^2 - \rho^2) |T|^s x_n \rightarrow 0$$

and

$$(|T(s, t)|^{rp} - \rho^{rp}) |T|^s x_n \rightarrow 0.$$

Similarly, we have

$$(|T(s, t)^*|^{rp} - \rho^{rp}) |T|^s x_n \rightarrow 0.$$

Hence

$$\begin{aligned} 0 &\leftarrow \langle (|T(s, t)|^{rp} - \rho^{rp}) |T|^s x_n, |T|^s x_n \rangle \\ &\geq \langle (|T|^{rp} - \rho^{rp}) |T|^s x_n, |T|^s x_n \rangle \\ &\geq \langle (|T(s, t)^*|^{rp} - \rho^{rp}) |T|^s x_n, |T|^s x_n \rangle \rightarrow 0. \end{aligned}$$

This implies

$$\langle (|T|^{rp} - \rho^{rp}) |T|^s x_n, |T|^s x_n \rangle \rightarrow 0.$$

Since $\frac{r}{2} \in (0, \min\{s, t\}]$, we have

$$\langle (|T|^{\frac{rp}{2}} - \rho^{\frac{rp}{2}}) |T|^s x_n, |T|^s x_n \rangle \rightarrow 0$$

by the same argument. Then

$$\begin{aligned} &\| (|T|^{\frac{rp}{2}} - \rho^{\frac{rp}{2}}) |T|^s x_n \|^2 \\ &= \langle (|T|^{\frac{rp}{2}} - \rho^{\frac{rp}{2}})^2 |T|^s x_n, |T|^s x_n \rangle \\ &= \langle (|T|^{rp} - \rho^{rp}) |T|^s x_n, |T|^s x_n \rangle - 2\rho^{\frac{rp}{2}} \langle (|T|^{\frac{rp}{2}} - \rho^{\frac{rp}{2}}) |T|^s x_n, |T|^s x_n \rangle \\ &\rightarrow 0. \end{aligned}$$

Hence

$$(|T|^{\frac{rp}{2}} - \rho^{\frac{rp}{2}}) |T|^s x_n \rightarrow 0$$

and so

$$(|T| - \rho) |T|^s x_n \rightarrow 0.$$

Then

$$|T|(|T| - \rho)x_n = |T|^{1-s}(|T| - \rho)|T|^s x_n \rightarrow 0.$$

Since

$$\begin{aligned} 0 &= \lim \langle |T|(|T| - \rho)x_n, x_n \rangle \\ &= \lim \| |T|x_n \|^2 - \rho \lim \langle |T|x_n, x_n \rangle \\ &= \lim \|Tx_n\|^2 - \rho \lim \langle Tx_n, x_n \rangle \\ &= \rho^2 - \rho \lim \langle Tx_n, x_n \rangle, \end{aligned}$$

we have

$$\langle |T|x_n, x_n \rangle \rightarrow \rho$$

and

$$\begin{aligned} \|(|T| - \rho)x_n\|^2 &= \| |T|x_n \|^2 - 2\rho \langle |T|x_n, x_n \rangle + \rho^2 \\ &\rightarrow \rho^2 - 2\rho^2 + \rho^2 = 0. \end{aligned}$$

This implies

$$(|T| - \rho)x_n \rightarrow 0.$$

Since

$$0 \leftarrow (T - \rho e^{i\theta})x_n = U(|T| - \rho)x_n + \rho(U - e^{i\theta})x_n$$

and $0 < \rho$, we have

$$(U - e^{i\theta})x_n \rightarrow 0.$$

Also,

$$\begin{aligned} \|(U - e^{i\theta})^*x_n\|^2 &= \|U^*x_n\|^2 - \langle U^*x_n, e^{-i\theta}x_n \rangle - \langle e^{-i\theta}x_n, U^*x_n \rangle + 1 \\ &\leq 1 - e^{i\theta}\langle x_n, Ux_n \rangle - e^{-i\theta}\langle Ux_n, x_n \rangle + 1 \\ &\leq -e^{i\theta}\langle x_n, (U - e^{i\theta})x_n \rangle - e^{-i\theta}\langle (U - e^{i\theta})x_n, x_n \rangle \rightarrow 0. \end{aligned}$$

Hence

$$(U - e^{i\theta})^*x_n \rightarrow 0$$

and

$$(T - \rho e^{i\theta})^*x_n = |T|(U - e^{i\theta})^*x_n + e^{-i\theta}(|T| - \rho)x_n \rightarrow 0.$$

□

Open problems

It is known that class A operator satisfies Putnam type inequality. However it is not known that Putnam type inequality holds for p - $wA(s, t)$ operators. It seems a difficult problem. We note some open problems for p - $wA(s, t)$ operators.

(1) M. Ito and T. Yamazaki [9] proved that $A(s, t)$ implies $wA(s, t)$. However it is not known whether p -class $A(s, t)$ implies p - $wA(s, t)$ for $0 < p < 1$ or not.

(2) It is known that if T is class $A(s, t)$ and $\mathcal{M} \subset \mathcal{H}$ is a T -invariant subspace, then $T|_{\mathcal{M}}$ is class $A(s, t)$. However it is not known whether this property holds for p - $wA(s, t)$ operator T .

(3) It is known that class A operator T is normaloid. But it is not known that p - $wA(s, t)$ operator T is normaloid or not.

REFERENCES

- [1] A. Aluthge, On p -hyponormal operators for $0 < p < 1$, Integral Equations Operator Theory. 13 (1990), 307 - 315.
- [2] M. Chō, M. Rashid, K. Tanahashi and A. Uchiyama, *Spectrum of class p - $wA(s, t)$ operators*, Acta Sci. Math. (Szeged), **82** (2016), 641–649.
- [3] B.P Duggal, Tensor Product of operators-strong stability and p -hyponormality, Glasgow Math. J 42.(200),371-381.
- [4] M. Fujii, D. Jung, S. H. Lee., M. Y. Lee., and R. Nakamoto, *Some classes of operators related to paranormal and log hyponormal operators*, Math. Japon., **51** (2000), 395–402.
- [5] T. Furuta, $A \geq B \geq O$ assures $(B^r A^p B^r)^{\frac{1}{q}} \geq B^{\frac{p+2r}{q}}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1 + 2r)q \geq (p + 2r)$, Proc. Amer. Math. Soc., **101** (1987), 85–88.
- [6] T. Furuta, *Generalized Aluthge transformation on p -hyponormal operators*, Proc. Amer. Math. Soc., **124** (1996), 3071–3075.
- [7] E. Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann., **123** (1951), 415–438.
- [8] M. Ito, *Some classes of operators associated with generalized Aluthge transformation*, SUT J. Math., **35** (1999), 149–165.

- [9] M. Ito and T. Yamazaki, *Relations between two inequalities $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{r}{2}}B^rA^{\frac{r}{2}})$ and their applications*, Integr. Equ. Oper. Theory, **44** (2002), 442–450.
- [10] I.H. Jeon and B.P. Duggal, *On operators with absolute value conditions*, J. Korean. Math. Soc **41**(2004), 617–627.
- [11] Jan Stochel., *Seminormality of operators from their tensor products*, Proc. Amer. Math. Soc **124**(1996) 435–440.
- [12] K. Löwner, *Über monotone Matrixfunktionen*, Math. Z., **38** (1934), 177–216.
- [13] K. Tanahashi and M. Chō, *Tensor Products of Log-hyponormal and of Class A (s, t) operators*, Glasgow Math J. **46**(2004) 91–91.
- [14] A. Uchiyama, K. Tanahashi and J. I. Lee, *Spectrum of class A (s, t) operators*, Acta Sci. Math. (Szeged), **70** (2004), 279–287.
- [15] M. Yanagida, *Powers of class $wA(s, t)$ operators with generalised Aluthge transformation*, J. Inequal. Appl., **7** (2002), 143–168.
- [16] T. Prasad and K. Tanahashi, *On class $p-wA(s, t)$ operators*, Functional Analysis, Approximation Computation, **6** (2) (2014), 39–42.
- [17] T. Yshino, *The p -hyponormality of the Aluthge transform*, Interdiscip. Inform. Sci., **3** (1997), 91–93.
- [18] J. Yuan and C. Yang, *Spectrum of class $wF(p, r, q)$ operators for $p+r \leq 1$ and $q \geq 1$* , Acta Sci. Math. (Szeged), **71** (2005), 767–779.
- [19] C. Yang and J. Yuan, *On class $wF(p, r, q)$ operators*, Acta. Math. Sci., **27** (2007), 769–780.

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